DEFORMATIONS OF LAGRANGIAN SUBVARIETIES OF HOLOMORPHIC SYMPLECTIC MANIFOLDS

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Abstract. We generalize Voisin's theorem on deformations of pairs of a symplectic manifold and a Lagrangian submanifold to the case of Lagrangian simple normal crossing subvarieties. We apply our results to the study of singular fibers of Lagrangian fibrations.

Contents

Introduction

In [Vo92] Voisin studied deformations of pairs $Y \subset X$ where X is an irreducible symplectic manifold and Y a complex Lagrangian submanifold. She showed that, roughly speaking, deformations of X where Y stays a complex submanifold are exactly those deformations, where Y stays Lagrangian.

We generalize Voisin's results to Lagrangian subvarieties with simple normal crossings.

Let M be the germ of the universal deformation space of X and denote by $\pi : \mathfrak{X} \longrightarrow M$ the universal family. By the Bogomolov-Tian-Todorov theorem, see [Bog78, Tia87, Tod89], we know that M is smooth. If the representative M is chosen simply connected, there is a canonical isomorphism $\alpha : R^2\pi_*\mathbb{C}_{\mathfrak{X}} \longrightarrow$

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$H^2(X, \mathbb{C})$ with the constant local system. Let $\omega \in R^2\pi_*\mathbb{C}_{\mathfrak{X}} \otimes \mathcal{O}_M$ be a class restricting to a symplectic form on the fibers of π. For a subvariety $i : Y \hookrightarrow X$ denote by M_i the germ of the universal deformation space for locally trivial deformations i and by $p : M_i \longrightarrow M$ the forgetful map. Then we have

Theorem 5.3. *Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, let $\nu : \widetilde{Y} \longrightarrow Y$ be the normalization and denote $j = i \circ \nu$. Consider the germs of the complex subspaces*

$$M_Y := \operatorname{im}(p : M_i \longrightarrow M) \text{ and } M'_Y := \{t \in M : j^*\alpha(\omega_t) = 0\}$$

of M. Then $M'_Y = M_Y$ and this space is smooth of codimension

$$\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y = \operatorname{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right)$$

in M.

Many of the intermediate steps in the proof of Theorem 5.3 are essentially as in [Vo92], but for the smoothness of M_Y we have to argue differently. For this, we develop ideas of Ran [Ra92Lif], [Ra92Def] by exploiting the interplay between deformation theory and Hodge theory. The necessary tools to apply Hodge theoretical arguments over an Artinian base are developed in [Le12]. As in [Vo92], we deduce the following

Corollary 5.4. *Let $K := \ker\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right)$, let q be the Beauville-Bogomolov quadratic form and consider the period domain*

$$Q := \{\alpha \in \mathbb{P}(H^2(X, \mathbb{C})) \mid q(\alpha) = 0, q(\alpha + \bar{\alpha}) > 0\}$$

of X. Then the period map $\wp : M \longrightarrow Q$ identifies M_Y with $\mathbb{P}(K) \cap Q$ locally at $[X] \in M$.

Let us spend some words about the structure of this article. In section 1 we recall the definition of *locally trivial* deformations. After recalling some facts about M and defining certain subspaces in section 2, we explain and adapt Voisin's results from [Vo92] in section 3. The spaces M_i and M_Y from Theorem 5.3 are treated in section 4. We develop Ran's ideas and explain the T^1-lifting principle to proof smoothness of M_i in case Y has simple normal crossings. Then, a variant of this principle enables us to deduce that the canonical map $p : M_i \longrightarrow M$ has constant rank in a neigbourhood of the distinguished points, which implies the smoothness of M_Y.
Furthermore, the projectivity of arbitrary Lagrangian subvarieties of an irreducible symplectic manifold is shown, see Corollary 4.5. This is used to

apply results from [Le12], but is also interesting in its own right. Again, the statement was known to Voisin in the smooth case.
Section 5 is finally puts together all previous theory to proof Theorem 5.3. We give applications to Lagrangian fibrations in section 6. Our results can be applied to most types of the general singular fibers of a Lagrangian fibration in the sense of Hwang-Oguiso [HO09].
The restriction to normal crossings comes from Proposition 4.10. The sheaf $\widetilde{\Omega}_Y$ determined there can be related to Hodge theory if Y has normal crossings. This is not only a technical condition, as easy examples already show.

Notations and conventions

We denote by k a field of characteristic zero. For a ring R we write $R[\varepsilon] := R[x]/x^2$ where $\varepsilon := x \mod (x^2)$. Set is the category of sets.
The term *algebraic variety* will stand for a separated reduced k-scheme of finite type. In particular, a variety may have several irreducible components. Similarly, a *complex variety* will be a separated reduced complex space. If there is no danger of confusion, we will drop the adjectives *algebraic* respectively *complex*. For an Artin ring R we do not distinguish between a quasi-coherent sheaf on $S = \operatorname{Spec} R$ and its R-module of global sections. A variety Y of equidimension n is called a *normal crossing variety* if for every closed point $y \in Y$ there is an $r \in \mathbb{N}_0$ such that $\widehat{\mathcal{O}}_{Y,y} \cong k[[y_1, \ldots, y_{n+1}]]/(y_1 \cdot \ldots \cdot y_r)$. It is called a *simple normal crossing variety* if in addition every irreducible component is nonsingular. For a regular function or, more generally, a section f of a coherent sheaf on a scheme X, we denote by $V(f)$ subscheme defined by the vanishing of f.
Let X be a scheme of finite type over $\mathbb{C}$. We write X^{an} for the complex space associated to X. For a quasi-coherent $\mathcal{O}_X$-module F we denote by F^{an} the associated $\mathcal{O}_{X^{\mathrm{an}}}$-module $\varphi^* F$ where $\varphi : X^{\mathrm{an}} \longrightarrow X$ is the canonical morphism of ringed spaces.

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1. Preliminaries

We recall basic definitions and results from deformation theory in the sense of Schlessinger [Sch68]. A detailed exposition is given in [Ser06], where most proofs of our statements are found or obtained by easy variations.

1.1. **Setup.** Let k be a fixed algebraically closed field. By Art_k we denote the category of local Artinian k-algebras with residue field k. The maximal ideal of an element $R \in \mathrm{Art}_k$ will be denoted by $\mathfrak{m}$. We write $\widehat{\mathrm{Art}}_k$ for the category of local noetherian k-algebras with residue field k, which are complete with respect to the $\mathfrak{m}$-adic topology. A *small extension* in Art_k is an exact sequence

$$0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0,$$

where $R' \longrightarrow R$ is a surjection in Art_k and $\mathfrak{m}'.J = 0$ for the maximal ideal $\mathfrak{m}'$ of R'. Because of this condition, the R'-module structure on J factors through $R'/\mathfrak{m}' = R/\mathfrak{m} = k$.

Definition 1.2. A *deformation functor* or *functor of Artin rings* is a functor $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ with $D(k) = \{\star\}$. The set $t_D = D(k[\varepsilon])$ is called the *tangent space* of D. A deformation functor D is said to be *prorepresentable* if there is a complete local noetherian k-algebra $R \in \widehat{\mathrm{Art}}_k$, such that $D \cong \mathrm{Hom}_k(R, \cdot)$.

Definition 1.3. If $D : \mathrm{Art}_k \longrightarrow \mathrm{Set}$ is a deformation functor, $R' \longrightarrow R$ is a morphism in Art_k and $\eta \in D(R)$ then we will write

$$D(R')_\eta := \varphi^{-1}(\eta) \subset D(R')$$

where $\varphi : D(R') \longrightarrow D(R)$ is the map induced by $R' \longrightarrow R$.

1.4. **Curvilinear extensions.** One can test smoothness by using only so-called *curvilinear* extensions. Namely, let R be a complete local noetherian k-algebra with maximal ideal $\mathfrak{m}$ and $A_n := k[t]/t^{n+1}$. Suppose R has the following lifting property for all $n \in \mathbb{N}$:

(1.1)
$$\begin{tikzcd} & A_{n+1} \arrow[d] \\ R \arrow[r] \arrow[ur, dashed, "\exists"] & A_n \end{tikzcd}$$

That is, for every k-algebra homomorphism $R \longrightarrow A_n$ there is a k-algebra homomorphism $R \longrightarrow A_{n+1}$ making (1.1) commutative. In this case we say

that R has the *curvilinear lifting property*. The following lemma is well-known, see [Le11, Lem I.1.6] for a proof.

Lemma 1.5. If R has the curvilinear lifting property, then R is a smooth k-algebra. $\square$

1.6. **Deformations of schemes.** Let X be an algebraic k-scheme. The functor

$$D_X : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{deformations of } X \text{ over } S = \operatorname{Spec} R\} \,/\sim$$

where $\sim$ is the relation of isomorphism, is called *functor of deformations of* X. It is proven as Corollary 2.6.4 in [Ser06] that for a smooth and projective k-scheme X with $H^0(X, T_X) = 0$, the functor D_X is prorepresentable. The proof there works for proper X as well.
Let $g : \mathcal{X} \longrightarrow S$ be a deformation of X over $S = \operatorname{Spec} R$. We put

$$T^1_{\mathcal{X}/R} := R^1 g_* T_{\mathcal{X}/S}, \quad T^1 := T^1_{X/k} = H^1(X, T_X). \tag{1.2}$$

As S is affine, $R^1 g_* T_{\mathcal{X}/S} \cong \check{H}^1(\mathcal{X}, T_{\mathcal{X}/S})$. By using the representation as a Čech-1-cocyle, one constructs a map $T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}$ and similar to [Ser06, Thm 2.4.1] one shows the following

Lemma 1.7. Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k. Assume that X is smooth over k. Then there is a natural isomorphism $T^1 \xrightarrow{\cong} t_{D_X}$. Moreover, the following holds. Let $\mathcal{X}' \longrightarrow S$ be a deformation of X over $S' = \operatorname{Spec} R'$ such that $\mathcal{X}' \times_{S'} S = \mathcal{X}$. Then the map $T^1_{\mathcal{X}/R} \longrightarrow D_X(R[\varepsilon])_{\mathcal{X}}$ is a bijection and the diagram

$$\begin{tikzcd} T^1_{\mathcal{X}'/R'} \arrow[r] \arrow[d] & T^1_{\mathcal{X}/R} \arrow[d] \\ D_X(R'[\varepsilon])_{\mathcal{X}'} \arrow[r] & D_X(R[\varepsilon])_{\mathcal{X}} \end{tikzcd}$$

is commutative, where we obtain $T^1_{\mathcal{X}'/R'} \longrightarrow T^1_{\mathcal{X}/R}$ by applying $R^1 g_*$ to the natural map $T_{\mathcal{X}'/S'} \longrightarrow T_{\mathcal{X}'/S'}$.

We call $T^1_{\mathcal{X}/R}$ a *relative tangent space* of D_X.

1.8. **Deformations of morphisms.** Let $i : Y \longrightarrow X$ be a morphism of algebraic k-schemes, let $R \in \mathrm{Art}_k$ and $S = \operatorname{Spec} R$, and let $I : \mathcal{Y} \longrightarrow \mathcal{X}$ be a deformation of i over S. It is called *(Zariski) locally trivial* if for every $x \in X$, $y \in Y$ with $i(y) = x$ there are open subsets $U \subset X$, $V \subset Y$ with

$y \in V$, $i(V) \subset U$ and an isomorphism

$$\begin{tikzcd}
\mathcal{X}_{|U} \arrow[rr, "\cong"] \arrow[dr] & & X_{|U} \times_k S \arrow[dl] \\
& S & \\
\mathcal{Y}_{|V} \arrow[rr, "\cong"] \arrow[ur] \arrow[uu, "I_{|V}"] & & Y_{|V} \times_k S \arrow[ul] \arrow[uu, "i_{|V} \times_k \mathrm{id}"']
\end{tikzcd}$$

In other words, $I : \mathcal{Y} \longrightarrow \mathcal{X}$ induces the trivial deformation on V and U. The functor

$$D_i^{\mathrm{lt}} : \mathrm{Art}_k \longrightarrow \mathrm{Set}, \quad R \mapsto \{\text{locally trivial deformations of } i \text{ over } S\} / \sim$$

where $\sim$ is the relation of isomorphism, is called the *functor of locally trivial deformations of* i.

1.9. **Sheaves controlling the deformations of a closed immersion.** Let $i : Y \hookrightarrow X$ be a closed immersion of algebraic k-schemes and suppose that X is smooth and proper and Y is a reduced locally complete intersection. Let $R \in \mathrm{Art}_k$, let $S = \operatorname{Spec} R$ and let

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\
& S
\end{tikzcd} \tag{1.3}$$

be a deformation of i. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{Y}$ in $\mathcal{X}$. By the hypothesis on Y, the sheaf $\mathcal{I}/\mathcal{I}^2$ is locally free and we have an exact sequence of sheaves on $\mathcal{Y}$

$$0 \longrightarrow T_{\mathcal{Y}/S} \longrightarrow T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{d^\vee} N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S} \longrightarrow 0, \tag{1.4}$$

where $N_{\mathcal{Y}/\mathcal{X}} := \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}})$. The sheaf $T^1_{\mathcal{Y}/S} := \operatorname{coker} d^\vee$ is supported on the singular locus of $\mathcal{Y} \longrightarrow S$. The *equisingular normal sheaf* is defined as

$$N'_{\mathcal{Y}/\mathcal{X}} := \ker(N_{\mathcal{Y}/\mathcal{X}} \longrightarrow T^1_{\mathcal{Y}/S}). \tag{1.5}$$

We define the sheaf T_I as the preimage of $T_{\mathcal{Y}/S}$ under the natural map $T_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}$ and obtain the exact sequence of sheaves on $\mathcal{X}$

$$0 \longrightarrow T_I \longrightarrow T_{\mathcal{X}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}} \longrightarrow 0. \tag{1.6}$$

The sheaf T_I is the relative version of the corresponding sheaf from [Ser06, 3.4.4]. It controls locally trivial deformations of a closed immersion in the sense of Lemma 1.10 below.

In [Ser06, Rem 3.4.18] it is shown that the functor D_i^{lt} is prorepresentable if X and Y are projective, X is smooth and $H^0(X, T_i) = 0$. As in the case of deformations of schemes, the proof carries over to proper schemes. Take $R \in \mathrm{Art}_k$, let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes, where Y is a reduced locally complete intersection and X is smooth over k. Let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd} \tag{1.7}$$

be a locally trivial deformation of i over $S = \operatorname{Spec} R$. As for deformations of schemes we introduce relative tangent spaces

$$T^1_{I/R} := R^1 g_* T_I, \qquad T^1 := T^1_{i/k} = H^1(X, T_i). \tag{1.8}$$

One constructs a natural map $T^1_{I/R} \longrightarrow D_i(R[\varepsilon])_I$, where $D_i(R[\varepsilon])_I$ is the fiber over I in the sense of Definition 1.3, similar as for deformations of schemes. As a straightforward generalization of [Ser06, Prop 3.4.17] we obtain

Lemma 1.10. Let $0 \longrightarrow J \longrightarrow R' \longrightarrow R \longrightarrow 0$ be a small extension in Art_k and let $i : Y \hookrightarrow X$ be a closed immersion of proper algebraic k-schemes where Y is a reduced locally complete intersection and X is smooth over k. Then there is a natural isomorphism $T^1 \xrightarrow{\cong} t_{D_i}$. Moreover, the following holds. Let I be as in (1.7), let $I' : \mathcal{Y}' \hookrightarrow \mathcal{X}'$ be a locally trivial deformation of i over R' such that $I' \times_{S'} S = I$ where $S' = \operatorname{Spec} R'$. Then the map $T^1_{I/R} \longrightarrow D_i(R[\varepsilon])_I$ is a bijection and the diagram

$$\begin{tikzcd} T^1_{I'/R'} \arrow[r] \arrow[d] & T^1_{I/R} \arrow[d] \\ D_i(R'[\varepsilon])_{I'} \arrow[r] & D_i(R[\varepsilon])_I \end{tikzcd}$$

is commutative, where we obtain $T^1_{I'/R'} \longrightarrow T^1_{I/R}$ by applying $R^1 g_*$ to the natural map $T_{I'} \longrightarrow T_I$.

Remark 1.11. The deformation functors D_i and D_X have their natural analogues in the category of complex spaces. Local triviality is defined using Euclidean instead of Zariski open sets. The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ induces a natural transformation between deformation functors in both categories. It is shown in [Le11, Lemma I.5.1] that this is an isomorphism of functors, which essentially follows from the fact that the functors have the same tangent and obstruction spaces.

2. Deformations of irreducible symplectic manifolds

Let X be an irreducible symplectic manifold, that is, a compact, simply connected Kähler manifold such that $H^0(X,\Omega_X^2) = \mathbb{C}\sigma$ for a symplectic form σ. In this section we review the universal deformation space M of X and discuss certain subspaces. As $H^0(X,T_X) = 0$ for irreducible symplectic manifolds, the Kuranishi family $\pi : \mathfrak{X} \longrightarrow M$ of X is universal at the point $0 \in M$ corresponding to X. Close to $0 \in M$ the fibers of π are again irreducible symplectic manifolds, see [Bea83, § 8]. M is known to be smooth by the Bogomolov-Tian-Todorov theorem [Bog78, Tia87, Tod89], see also [GHJ, Thm 14.10] for an introduction.

2.1. **Hodge bundles and the Gauß-Manin connection.** Consider the vector bundle $\mathscr{H}^k$ on M given by

$$\mathscr{H}^k := R^k\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \otimes \mathcal{O}_M.$$

It is filtered by subbundles $\mathscr{F}^p\mathscr{H}^k$ of $\mathscr{H}^k$ with fiber $\left(\mathscr{F}^p\mathscr{H}^k\right)_t = F^pH^k(X_t)$ at $t \in M$, the Hodge filtration on $H^k(X_t)$. We define the bundles

$$\mathscr{H}^{p,q} := \mathscr{F}^p\mathscr{H}^{p+q}/\mathscr{F}^{p+1}\mathscr{H}^{p+q}$$

The fiber of $\mathscr{H}^{p,q}$ at $t \in M$ is canonically identified with $H^q(X_t,\Omega^p_{X_t})$. There is a local system $\mathscr{H}^k_{\mathbb{C}} := R^k\pi_*\underline{\mathbb{C}}_{\mathfrak{X}} \hookrightarrow \mathscr{H}^k$ and the associated flat connection $\nabla : \mathscr{H}^k \longrightarrow \mathscr{H}^k \otimes \Omega_M$ is called the *Gauß-Manin* connection. It fulfills the so-called *Griffiths transversality* $\nabla\left(\mathscr{F}^p\mathscr{H}^k\right) \subset \mathscr{F}^{p-1}\mathscr{H}^k \otimes \Omega_M$. Therefore, it induces morphisms $\bar\nabla_p : \mathrm{Gr}^p_{\mathscr{F}}\mathscr{H}^k \longrightarrow \mathrm{Gr}^{p-1}_{\mathscr{F}}\mathscr{H}^k \otimes \Omega_M$ between the graded objects of the filtration. These maps are $\mathcal{O}_M$-linear and therefore corresponds to a map $\bar\nabla_p : \mathrm{Gr}^p_{\mathscr{F}}\mathscr{H}^k \longrightarrow \mathrm{Hom}(T_M, \mathrm{Gr}^{p-1}_{\mathscr{F}}\mathscr{H}^k)$. By a theorem of Griffiths its fiber at the point $t \in M$ can be identified with the map

$$H^{k-p}(X_t,\Omega^p_{X_t}) \longrightarrow \mathrm{Hom}\left(H^1(X_t,T_{X_t}), H^{k-p-1}(X_t,\Omega^p_{X_t})\right) \tag{2.1}$$

given by cup-product and contraction.

2.2. **Hodge loci.** Let $\beta \in H^k(X,\mathbb{C})$ be a cohomology class of type (p,q) with respect to the Hodge decomposition $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$. Suppose that M is simply connected. Then the local system $\mathscr{H}^k_{\mathbb{C}}$ is trivial and β extends to a global section of $\mathscr{H}^k_{\mathbb{C}}$, that is, a flat section of $\mathscr{H}^k$, which we also denot by β. We write β_t for its fiber at t. The following definition and some basic properties can be found in [Vo2, Ch 5.3].

Definition 2.3. The *Hodge locus* associated to β is the complex subspace $M_\beta \hookrightarrow M$ defined by the vanishing of the induced section

$$\bar\beta : \mathcal{O}_M \longrightarrow \mathscr{H}^k \longrightarrow \mathscr{H}^k/\mathscr{F}^p\mathscr{H}^k.$$

So the Hodge locus M_β is the locus of all $t \in M$, where $\beta_t \in F^pH^k(X_t)$. If β is an integral or at least real cohomology class of Hodge type (p,p), then

$$M_\beta = \{t \in M \mid \beta_t \in H^{p,p}(X_t)\} \tag{2.2}$$

as β is fixed under complex conjugation and $F^pH^{2p}(X_t) \cap \overline{F^pH^{2p}(X_t)} = H^{p,p}(X_t)$.

2.4. **Subspaces of M associated with Lagrangian subvarieties.** Let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian subvariety in an irreducible symplectic manifold X of dimension $2n$. Let M be a simply connected representative of the universal deformation space of X, let $0 \in M$ be the point corresponding to X and let $\pi : \mathfrak{X} \longrightarrow M$ be the universal family. Following Voisin [Vo92], we define three subspaces of M associated to Y.

We take a relative symplectic form $\omega \in R^0\pi_*\Omega^2_{\mathfrak{X}/M} \hookrightarrow R^2\pi_*\mathbb{C}_{\mathfrak{X}} \otimes \mathcal{O}_M$ and write $\omega_t := \omega|_{X_t}$ for the symplectic form on the fiber $X_t = \pi^{-1}(t)$. If the representative M is chosen simply connected, there is a canonical isomorphism $\alpha : R^2\pi_*\mathbb{C}_{\mathfrak{X}} \longrightarrow H^2(X,\mathbb{C})$ with the constant local system. We denote by $\nu : \widetilde{Y} \longrightarrow Y$ a resolution of singularities and by $j = i \circ \nu$ the composition.

Definition 2.5. We define $M'_Y := V(j^*\alpha(\omega))$. In other words,

$$M'_Y = \left\{ t \in M \mid j^*\alpha(\omega_t) = 0 \text{ in } H^2(\widetilde{Y},\mathbb{C}) \right\}. \tag{2.3}$$

The Lagrange property of Y implies $0 \in M'_Y$. Clearly, this definition is independent of the resolution $\nu : \widetilde{Y} \longrightarrow Y$.

If $[Y] \in H^{2n}(X,\mathbb{Z})$ denotes the Poincaré dual of the fundamental cycle of Y, we write μ_0 for the map $H^2(X,\mathbb{C}) \longrightarrow H^{2+2n}(X,\mathbb{C})$ given by cup product with $[Y]$. This map is a morphism of Hodge structures and can be factored as

$$\mu_0 : H^2(X,\mathbb{C}) \xrightarrow{j^*} H^2(\widetilde{Y},\mathbb{C}) \xrightarrow{j_*} H^{2+2n}(X,\mathbb{C}).$$

By lifting $[Y]$ to a flat section of $\mathscr{H}^2$, we can extend μ_0 to a map $\mu : \mathscr{H}^2 \longrightarrow \mathscr{H}^{2+2n}$. Consider the section $\mu \circ \omega \in H^0(M, \mathscr{H}^{2+2n})$ where ω is the relative symplectic form.

Definition 2.6. We put $M'_{[Y]} := V(\mu \circ \omega)$. In other words,

$$M'_{[Y]} = \{t \in M \mid \mu(\omega_t) = 0\} = \{t \in M \mid [Y]_t \cup \omega_t = 0\}. \tag{2.4}$$

The Lagrange property ensures that $0 \in M'_{[Y]}$.

Finally, we denote by $M_{[Y]}$ the Hodge locus associated to the class $[Y]$ of Y in $H^{2n}(X,\mathbb{C})$, see section 2.2. As $[Y]$ is integral and of type (n,n), its Hodge locus is set-theoretically given by

$$M_{[Y]} = \{t \in M \mid [Y]_t \in H^{n,n}(X_t)\}, \tag{2.5}$$

where as above $[Y]_t$ is the restriction to the fiber over t of the unique flat section of $\mathscr{H}^{2n}$ extending $[Y]$. In particular, we have $0 \in M_{[Y]}$.

Remark 2.7. Observe that the spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ may be defined for arbitrary subvarieties $Y \hookrightarrow X$. Singularities do not cause any harm, as $M'_{[Y]}$ and $M_{[Y]}$ only depend on the class $[Y]$ and M'_Y is defined via a resolution of singularities. As we are only interested in the germs at 0 of these subspaces, we may and will assume that M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ are connected.
Let us collect some simple observations on the relation among the spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$. As $\mu = j^* j_*$ we have $M'_Y \subset M'_{[Y]}$. If $Y = \cup_i Y_i$ is a decomposition into irreducible components, then $M'_Y = \cap_i M'_{Y_i}$ as a direct consequence of the definitions. Moreover, the inclusions $M'_{[Y]} \supset \cap_i M'_{[Y_i]}$ and $M_{[Y]} \supset \cap_i M_{[Y_i]}$ are immediate.

3. Voisin's results adapted

Essentially everything in this section is taken from [Vo92], but with some slight modifications to our situation. So unless the contrary is explicitly stated, all results presented are Voisin's. We will freely use the notations of section 2.

Proposition 3.1. $M_{[Y]} = M'_{[Y]}$ as sets.

Proof. We first show $M'_{[Y]} \subset M_{[Y]}$. We write $[Y]_t = \sum_{p+q=2n} [Y]_t^{p,q}$ with respect to the Hodge decomposition at $t \in M'_{[Y]}$. We want to show that $[Y]_t = [Y]_t^{n,n}$. As $[Y]$ is integral, we have $\overline{[Y]_t^{p,q}} = [Y]_t^{q,p}$ and so it suffices to show that $[Y]_t^{p,q} = 0$ for $p < n$. As ω_t is of type $(2,0)$ on X_t the assumption $\mu(\omega_t) = 0$ gives $\omega_t \cup [Y]_t^{p,q} = 0$ for all p, q. But $\omega_t^k \cup : \Omega_{X_t}^{n-k} \longrightarrow \Omega_{X_t}^{n+k}$ is an isomorphism for $k \geq 0$, which can be seen pointwise by linear algebra. Hence the map $\omega_t \cup$ is injective for $p < n$, which yields that $[Y]_t^{p,q} = 0$ for $p < n$, as needed.
For the inclusion $M_{[Y]} \subset M'_{[Y]}$ it suffices to show that $M_{[Y]} \cap M'_{[Y]}$ is non-empty and open in $M_{[Y]}$ as it is automatically closed and we may assume that $M_{[Y]}$ is connected, see Remark 2.7. This is the only point where we use that Y is Lagrangian, namely for the nonemptiness. For $t \in M_{[Y]}$ the morphism $\mu : H^2(X_t, \mathbb{C}) \longrightarrow H^{2n+2}(X_t, \mathbb{C})$ is a morphism of Hodge structures of degree (n, n) and hence gives morphisms $\mu^{p,q} : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{p+n,q+n}$ for $p + q = 2$. By semi-continuity they satisfy $\mathrm{rk}\mu^{p,q}(t') \geq \mathrm{rk}\mu^{p,q}(t)$ for all t' in a small neighborhood U of t. As $\mu = \mu^{2,0} \oplus \mu^{1,1} \oplus \mu^{0,2}$ as a C^∞-morphism on U, the rank of the summands remains constant in t. So as for $t = 0 \in M_{[Y]} \cap M'_{[Y]}$ we have $\mu^{2,0} = 0 = \mu^{0,2}$ this remains true in a neigbourhood and so the claim follows. □

Proposition 3.2. The varieties $M_{[Y]}$ and ${M'}_{[Y]}$ are smooth near $t=0$ and their codimension in M is $r_{[Y]} = \mathrm{rk}\left(\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2n+2}(X,\mathbb{C})\right)$. In particular, $M_{[Y]} = {M'}_{[Y]}$ as varieties by the preceding proposition.

Proof. We argue only for ${M'}_{[Y]}$, the case of $M_{[Y]}$ is similar. Consider the sheaf $\mathscr{H}_\mu := \mu(\mathscr{H}^2) \subset \mathscr{H}^{2n+2}$. As μ is defined on the level of local systems its rank is locally constant, so this is a vector bundle of rank $r_{[Y]}$. The variety ${M'}_{[Y]}$ is defined by the vanishing of the section $\mu(\omega) \in \mathscr{H}_\mu$, hence $\operatorname{codim} {M'}_{[Y]} \le r_{[Y]}$. So it suffices to show that the rank of the system of equations $\mu(\omega) = 0$ is equal to $r_{[Y]}$. Recall that the Gauß-Manin connection is given by the differential d if we trivialize with flat sections. This implies that for μ to have rank $r_{[Y]}$ at 0, the classes $\bar{\nabla}_{\chi,0}(\mu_0(\omega_0))$ for $\chi \in T_{M,0} = H^1(X,T_X)$ have to span a vector space of dimension $r_{[Y]}$.
We have $\nabla_\chi(\mu(\omega_t)) = \mu(\nabla_\chi \omega_t)$ and by (2.1) the Gauß-Manin connection $\bar{\nabla} : \mathscr{F}^2\mathscr{H}^2 \longrightarrow \mathrm{Hom}(T_M, \mathscr{F}^1\mathscr{H}^2/\mathscr{F}^2\mathscr{H}^2)$ at t is identified with the morphism

$$H^0(\Omega^2_{X_t}) \longrightarrow \mathrm{Hom}\left(H^1(T_{X_t}), H^1(\Omega_{X_t})\right)$$

given by the cup product and contraction. As ω_0 is non-degenerate and of type $(2,0)$ the $\nabla_\chi \omega_t$ span the whole of $H^{1,1}(X)$ at $t=0$. □

Lemma 3.3. The tangent space of M'_Y at 0 is given by

$$T_{M'_Y,0} = \ker\left(j^* \circ \omega' : H^1(X,T_X) \xrightarrow{\omega'} H^1(X,\Omega_X) \xrightarrow{j^*} H^1(\widetilde{Y},\Omega_{\widetilde{Y}})\right) \tag{3.1}$$

where ω' is the isomorphism induced by the symplectic form on X.

Proof. Locally at $0 \in M$ the space M'_Y is cut out by the equation $j_t^*\omega_t = 0$. Therefore the tangent space at 0 is given by the equation

$$0 = (\nabla j_t^* \omega_t)\,|_{t=0} = j^*\,(\nabla \omega_t)\,|_{t=0}.$$

The Gauß-Manin conection at 0 can be identified with the map

$$H^0(X,\Omega^2_X) \longrightarrow \mathrm{Hom}(H^1(X,T_X), H^1(X,\Omega_X)), \quad \psi \mapsto (u \mapsto \psi(u))$$

given by cup product and contraction, which concludes the proof. □

Lemma 3.4. Let X be an irreducible symplectic manifold of dimension $\dim X = 2n$. Let $Y \subset X$ be an irreducible Lagrangian subvariety, let $\nu : \widetilde{Y} \longrightarrow Y$ a resolution of singularities and put $j = i \circ \nu$. Then

$$\ker\left(\mu : H^2(X,\mathbb{C}) \longrightarrow H^{2n+2}(X,\mathbb{C})\right) = \ker\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right).$$

Proof. We show equality of the respective kernels with real coefficients. From $\mu = j_* j^*$ we immediately have $\ker j^* \subset \ker \mu$. For the other inclusion we choose a Kähler class $\kappa \in H^2(X,\mathbb{R})$. We have to show that j_* is injective on $\operatorname{im} j^*$.

Assume $n = 1$. As $\widetilde{Y}$ is connected, $H^2(\widetilde{Y}, \mathbb{C}) \cong \mathbb{C}$ and the map $j_* : H^2(\widetilde{Y}, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C})$ is given by $1 \mapsto [Y]$. As X is Kähler, $[Y] \neq 0$. So j_* is injective and the claim follows.

If $n \geq 2$, choose a Kähler class $\kappa \in H^2(X, \mathbb{R})$. For each $Y' \longrightarrow \widetilde{Y}$ with non-singular Y' the induced map $H^2(\widetilde{Y}, \mathbb{C}) \longrightarrow H^2(Y', \mathbb{C})$ is injective, see for example [BHPV, Chapter I, Topology and Algebra 1., (1.2) Corollary, p.11]. Every resolution is dominated by a resolution $Y' \longrightarrow Y$ fitting into a diagram

$$\begin{tikzcd} \widetilde{Y} \arrow[dr] & Y' \arrow[l] \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ & Y \arrow[r, hook] & X \end{tikzcd}$$

where X' is obtained by a sequence of blow-ups of X in smooth centers. Thus, we may assume that $\widetilde{Y} = Y'$ is such a resolution. Hence there is a Kähler class of the form $\widetilde{\kappa} = j^*\kappa - \sum_i \delta_i E_i \in H^2(\widetilde{Y}, \mathbb{R})$ where the E_i are exceptional divisors and $\delta_i \in \mathbb{Q}$ are positive. We define a bilinear form

$$q(\alpha, \beta) := \int_{\widetilde{Y}} \widetilde{\kappa}^{n-2}.\alpha.\beta \qquad \alpha, \beta \in H^2(\widetilde{Y}, \mathbb{C})$$

on $H^2(\widetilde{Y}, \mathbb{C})$. For $\alpha, \beta \in H^2(X, \mathbb{R})$ this gives

$$\begin{aligned} q(j^*\alpha, j^*\beta) &= \int_{\widetilde{Y}} \widetilde{\kappa}^{n-2}.j^*(\alpha.\beta) = \int_X j_* \left(\widetilde{\kappa}^{n-2}.j^*(\alpha.\beta)\right) \\ &= \int_X j_* \left(\widetilde{\kappa}^{n-2}.j^*(\alpha.\beta)\right) = \int_X j_*(\widetilde{\kappa}^{n-2}).\alpha.\beta \\ &= \int_X \mu(\kappa^{n-2}).\alpha.\beta = \int_X \kappa^{n-2}.\mu(\alpha).\beta. \end{aligned}$$

Here we used that $j_* E_i = 0$ or equivalently $j_*\widetilde{\kappa} = \kappa \cup [Y] = \mu(\kappa)$. From the calculation we see that if $\mu(\alpha) = 0$, then $q(j^*\alpha, j^*\beta) = 0$ for all $\beta \in H^2(X, \mathbb{R})$. To conclude that $j^*\alpha = 0$ it would be sufficient to see that q is non-degenerate on $\operatorname{im} j^* \subset H^2(\widetilde{Y}, \mathbb{R})$. On the whole of $H^2(\widetilde{Y}, \mathbb{R})$ the form q is non-degenerate by the Hodge index theorem, see [Vo1, Thm 6.33]. Here we need that $\widetilde{\kappa}$ is a Kähler class. That q remains non-degenerate on the subspace $\operatorname{im} j^*$ can also be deduced as follows. As we have seen $\operatorname{im} j^* \subset H^{1,1}(\widetilde{Y}, \mathbb{R}) := H^{1,1}(\widetilde{Y}) \cap H^2(\widetilde{Y}, \mathbb{R})$ and on $H^{1,1}(\widetilde{Y}, \mathbb{R})$ the form q is non degenerate and has signature $(1, h^{1,1} - 1)$. We know that $q(j^*\kappa, j^*\kappa) > 0$ and so q is negative definite on $j^*\kappa^\perp$. Write $j^*\alpha = c \cdot j^*\kappa + \alpha'$ where $\alpha' \in j^*\kappa^\perp$. The decomposition shows that $\alpha' \in \operatorname{im} j^*$ as well. Then if $j^*\alpha \neq 0$ at least one of the numbers $q(j^*\alpha, j^*\kappa)$, $q(j^*\alpha, \alpha')$ is nonzero and so $\mu(\alpha) \neq 0$ completing the proof. $\square$

Corollary 3.5. Let X be an irreducible symplectic manifold, let $Y \subset X$ be an irreducible Lagrangian subvariety with normal crossing singularities. Then we have $M'_{[Y]} = M'_Y$. In particular, M'_Y is smooth at 0.

Proof. We observed that $M'_Y \subset M'_{[Y]}$ in Remark 2.7. As $M'_{[Y]}$ is smooth by Proposition 3.2 it suffices to show that $M'_Y \supset M'_{[Y]}$ holds set-theoretically. By definition $t \in M'_{[Y]}$ if $\omega_t \cup [Y]_t = 0$ and $t \in M'_Y$ if $j_t^*\omega_t = 0$. But $\omega_t \cup [Y]_t = 0$ if and only if $j_t^*\omega_t = 0$ by Lemma 3.4. □

4. Deformations of Lagrangian subvarieties

Let X be an irreducible symplectic manifold and let $i : Y \hookrightarrow X$ be the inclusion of a Lagrangian simple normal crossing subvariety. In this section, we will proof smoothness of the space M_i of locally trivial deformations of i and the statement about factorisation of $p : M_i \longrightarrow M$ made in the introduction.

The proofs are elaborations of Ran's ideas [Ra92Lif], [Ra92Def] and the method is related to the T^1-lifting principle. These smoothness results play an important role in the proof of our main result, Theorem 5.3. Sections 4 and 5 rely heavily on results of [Le12], where the Hodge theory for locally trivial deformations of normal crossing varieties was studied. In particular, we would like to recall the following definition.

Definition 4.1. Let $R \in \mathrm{Art}_k$ and let $f : \mathcal{Y} \longrightarrow S = \operatorname{Spec} R$ be a locally trivial deformation of a k-variety Y. We define $\tau^k_{\mathcal{Y}/S} \subset \Omega^k_{\mathcal{Y}/S}$ to be the subsheaf of sections whose support is contained in the singular locus of f. We put $\widetilde{\Omega}^k_{\mathcal{Y}/S} := \Omega^k_{\mathcal{Y}/S}/\tau^k_{\mathcal{Y}/S}$.

4.2. **Projectivity of Lagrangian subvarieties.** If $Y \subset X$ is a smooth Lagrangian subvariety, then by an argument of Voisin, Y is projective even if X is only Kähler, see [Cam06, Prop 2.1]. If $Y \subset X$ is a singular Lagrangian subvariety, it is natural to ask whether Y is still projective. Later on we will use that Y is an algebraic variety or more precisely, that $Y = \mathfrak{Y}^{\mathrm{an}}$ for an algebraic variety $\mathfrak{Y}$. We have

Lemma 4.3. Let $i : Y \hookrightarrow X$ be a complex Lagrangian subvariety in an irreducible symplectic manifold. There is a line bundle L on Y such that $\mathrm{c}_1(L) = i^*\lambda$ for some Kähler class λ on X.

Proof. Isomorphism classes of line bundles on Y are classified by the group $H^1(Y, \mathcal{O}_Y^\times)$, see [GR77, Kap V, § 3.2]. This cohomology group appears in

the commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & H^1(Y,\mathcal{O}_Y^\times) \arrow[r] & H^2(Y,\mathbb{Z}) \arrow[r] \arrow[d] & H^2(Y,\mathcal{O}_Y) \arrow[r] & \dots \\ & & H^2(Y,\mathbb{C}) \arrow[r] & \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y) \arrow[u] & \\ & & & \mathbb{H}^2(Y,\widetilde{\Omega}^{\geq 1}_Y) \arrow[u] & \end{tikzcd}$$

where the first line is the long exact sequence associated to the exponential sequence, see [GR77, Kap V, § 2.4], and the right vertical column comes from the short exact sequence

$$0 \longrightarrow \widetilde{\Omega}^{\geq 1}_Y \longrightarrow \widetilde{\Omega}^\bullet_Y \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Here we need that $\widetilde{\Omega}^0_Y = \mathcal{O}_Y$. This is true, as Y is reduced, because then Y does not have embedded points. To obtain a holomorphic line bundle L on Y it is sufficient to find a class $\alpha \in H^2(Y,\mathbb{Z})$, such that the image in $\mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y)$ comes from $\mathbb{H}^2(Y,\widetilde{\Omega}^{\geq 1}_Y)$. Such L will have $\mathrm{c}_1(L) = \alpha$.
Let $H_X := \mathrm{im}(i^* : \mathbb{H}^2(X,\Omega^\bullet_X) \longrightarrow \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y))$ where $i : Y \hookrightarrow X$ is the inclusion. From the spectral sequence for $\Omega^\bullet$ we obtain maps

$$\begin{tikzcd} H^0(X,\Omega^2_X) \arrow[r] \arrow[d] & \mathbb{H}^2(X,\Omega^\bullet_X) \arrow[d, "i^*"] & H^2(X,\mathbb{C}) \arrow[l, "\cong"'] \arrow[d] \\ H^0(Y,\widetilde{\Omega}^2_Y) \arrow[r] & \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y) & H^2(Y,\mathbb{C}) \arrow[l] \end{tikzcd}$$

As Y is Lagrangian and by definition $\widetilde{\Omega}^2_Y$ is torsion free we have $i^*\omega = 0$ in $H^0(Y,\widetilde{\Omega}^2_Y)$ where $\omega \in H^0(X,\Omega^2_X)$ is the symplectic form on X. By Hodge-decomposition $\mathbb{H}^2(X,\Omega^\bullet_X) \cong H^2(X,\mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ and by Dolbeault's theorem $H^0(X,\Omega^2_X) \cong H^{2,0}(X)$ we see that $H^{2,0}(X) \cong \mathbb{C}\omega$ maps to zero under i^*. From the left square of the above diagram, we see that also the complex conjugate $H^{0,2}(X) \cong \mathbb{C}\overline{\omega}$ maps to zero, as the map $H^2(X,\mathbb{C}) \longrightarrow H^2(Y,\mathbb{C})$ is defined over $\mathbb{R}$. Thus

$$\begin{aligned} H_X &= \mathrm{im}(i^* : H^2(X,\mathbb{C}) \longrightarrow \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y)) \\ &= \mathrm{im}(i^* : H^{1,1}(X) \longrightarrow \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y)). \end{aligned} \tag{4.1}$$

Let $H_{X,\mathbb{R}} = \mathrm{im}(i^* : H^2(X,\mathbb{R}) \longrightarrow \mathbb{H}^2(Y,\widetilde{\Omega}^\bullet_Y))$. The last description in (4.1) implies that $i^*(\mathcal{K}_X)$ is open in $H_{X,\mathbb{R}}$ where $\mathcal{K}_X$ is the Kähler cone of X. Indeed, $\mathcal{K}_X$ is open in $H^{1,1}(X)_\mathbb{R} = H^{1,1}(X) \cap H^2(X,\mathbb{R})$ and the map $H^{1,1}(X) \longrightarrow H_X$ is surjective. Therefore, also $H^{1,1}(X)_\mathbb{R} \longrightarrow H_{X,\mathbb{R}}$ is surjective so that $i^*(\mathcal{K}_X)$ is open in $H_{X,\mathbb{R}}$. We show next that $i^*(\mathcal{K}_X)$ meets the

image of $H^2(Y, \mathbb{Z})$. Let us consider

$$H_{X,\mathbb{Q}} = \operatorname{im}(i^* : H^2(X, \mathbb{Q}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}_Y^\bullet)) \subset H_X.$$

This is dense in $H_{X,\mathbb{R}}$ as $H^2(X, \mathbb{Q})$ is dense in $H^2(X, \mathbb{R})$ and so it meets $i^*(\mathcal{K}_X)$, say in $\alpha' \in H_{X,\mathbb{Q}} \cap i^*(\mathcal{K}_X)$. Then a multiple $\alpha = m \cdot \alpha'$ is contained in $\operatorname{im}(H^2(X, \mathbb{Z}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}_Y^\bullet)) \cap i^*\mathcal{K}_X$ and we obtain a line bundle L on Y with the desired property by using the exponential sequence as explained above. □

Remark 4.4. The only difference to Voisin's original proof is that we have to be careful with the fact that $H^2(Y, \mathbb{C}) \longrightarrow \mathbb{H}^2(Y, \widetilde{\Omega}_Y^\bullet)$ is not an isomorphism.

Corollary 4.5. If $Y \subset X$ is a complex Lagrangian subvariety in an irreducible symplectic manifold, then Y is a projective algebraic variety.

Proof. By the preceding lemma, there is a line bundle L on Y whose first Chern class is the restriction of some Kähler class of X. Then Y is projective by [GPR94, Chapter V, Corollary 4.5], see also [Gra62, 3, Satz 1 and Satz 2]. □

4.6. **Deformations of Lagrangian subvarieties.** Suppose $g : \mathcal{X} \longrightarrow S$ is a deformation of an irreducible symplectic manifold X over $S = \operatorname{Spec} R$ for $R \in \operatorname{Art}_k$. The symplectic form ω_0 on X extends to an everywhere non-degenerate section $\omega \in R^0 g_* \Omega^2_{\mathcal{X}/S}$, as this module is free.

Lemma 4.7. Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety. If $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ is a locally trivial deformation of i over S, then $\mathcal{Y}$ is Lagrangian with respect to the symplectic form ω on $\mathcal{X}$.

Proof. Let $\tilde{f} : \widetilde{\mathcal{Y}} \longrightarrow S$ be the locally trivial deformation of the normalization of Y obtained from [Le12, Lemma 4.5]. Note that Y is projective by Corollary 4.5, so Lemma [Le12, Lemma 4.5] can be applied. As Y has simple normal crossings, $f \circ \nu : \widetilde{\mathcal{Y}} \longrightarrow S$ is smooth and the restriction $R^0 g_* \Omega^2_{\mathcal{X}/S} \xrightarrow{j^* := \nu^* \circ i^*} R^0 f_* \Omega^2_{\widetilde{\mathcal{Y}}/S}$ has constant rank by [Le12, Prop. 4.18]. As $\operatorname{rk}(j^* \otimes \mathbb{C}) = 0$ on the central fiber, j^* is identically zero and thus $\mathcal{Y}$ is Lagrangian. □

Lemma 4.8. Let $i : Y \hookrightarrow X$ be a locally complete intersection Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \operatorname{Spec} R$ where $R \in \operatorname{Art}_{\mathbb{C}}$ and let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd} \tag{4.2}$$

be a locally trivial deformation of i over S. Then the symplectic form $\omega \in R^0 g_* \Omega^2_{\mathcal{X}/S}$ induces a morphism between the exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{I}/\mathcal{I}^2 \arrow[r] \arrow[d, dashed, "\omega^{-1}"] & \Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] \arrow[d, "\omega^{-1}"] & \Omega_{\mathcal{Y}/S} \arrow[r] \arrow[d, dashed, "\omega'"] & 0 & \\
0 \arrow[r] & T_{\mathcal{Y}/S} \arrow[r] & T_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}} \arrow[r] & N_{\mathcal{Y}/\mathcal{X}} \arrow[r] & T^1_{\mathcal{Y}/S} \arrow[r] & 0.
\end{tikzcd}
\tag{4.3}
$$

Proof. Since ω is non-degenerate, the map $\omega^{-1} : \Omega_{\mathcal{X}/S} \longrightarrow T_{\mathcal{X}/S}$ is an isomorphism. This will induce the other morphisms in the diagram as explained below. The composition $\varphi : \mathcal{I}/\mathcal{I}^2 \longrightarrow N_{\mathcal{Y}/\mathcal{X}} = \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_{\mathcal{Y}})$ is zero at smooth points. So $M := \mathrm{im}\,\varphi$ is torsion. But $\mathcal{Y}$ is a locally complete intersection, so $\mathcal{I}/\mathcal{I}^2$ is locally free and by [Mat80, 16, Thm 30] the submodule M is zero. So the restriction of ω^{-1} to $\mathcal{I}/\mathcal{I}^2$ factors through $T_{\mathcal{Y}/S}$. Once we have this, we obtain a morphism $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$, as the first line of (4.3) is exact, by lifting sections to $\Omega_{\mathcal{X}/S} \otimes \mathcal{O}_{\mathcal{Y}}$. $\square$

Corollary 4.9. If in the situation of the preceding lemma the morphism $f : \mathcal{Y} \longrightarrow S$ is smooth, then ω gives an isomorphism $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$.

Proof. As f is smooth, $T^1_{\mathcal{Y}/S} = 0$. So (4.3) gives a surjection $\omega : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$. As both $\Omega_{\mathcal{Y}/S}$ and $N_{\mathcal{Y}/\mathcal{X}}$ are locally free, the claim follows. $\square$

Note that $\mathcal{I}/\mathcal{I}^2 \longrightarrow T_{\mathcal{Y}/S}$ is not in general an isomorphism as $\Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$ might have a kernel. The following Proposition determines this kernel.

Proposition 4.10. Let $i : Y \hookrightarrow X$ be a locally complete intersection Lagrangian subvariety in an irreducible symplectic manifold X, let $S = \mathrm{Spec}\, R$ where $R \in \mathrm{Art}_{\mathbb{C}}$ and let $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over S as in (4.2). Let $\omega' : \Omega_{\mathcal{Y}/S} \longrightarrow N_{\mathcal{Y}/\mathcal{X}}$ be as in (4.3) and let $N'_{\mathcal{Y}/\mathcal{X}}$ be the equisingular normal sheaf defined in (1.5). Then the diagram

$$
\begin{tikzcd}
\Omega_{\mathcal{Y}/S} \arrow[r, "\omega"] \arrow[d] & N_{\mathcal{Y}/\mathcal{X}} \\
\widetilde{\Omega}_{\mathcal{Y}/S} \arrow[r, dashed, "\exists \widetilde{\omega}"'] & N'_{\mathcal{Y}/\mathcal{X}} \arrow[u, hook]
\end{tikzcd}
\tag{4.4}
$$

can be completed and $\widetilde{\omega} : \widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow N'_{\mathcal{Y}/\mathcal{X}}$ is an isomorphism. The analogue is true in the analytic setting.

Proof. As $\mathcal{Y}$ is a locally complete intersection, $N_{\mathcal{Y}/\mathcal{X}}$ is locally free, hence Cohen-Macaulay. Therefore it has no embedded primes by [Mat80, 16, Thm 30], hence $\tau^1_{\mathcal{Y}/S}$ maps to zero and $\widetilde{\omega}$ exists. But as ω is an isomorphism at

smooth points of f, the support of $\ker\omega$ is contained in the singular locus of f, hence $\ker\omega\subset\tau^k_{\mathcal{Y}/S}$ and $\widetilde{\omega}$ is injective. Moreover, $\widetilde{\Omega}_{\mathcal{Y}/S}$ maps onto $\ker(N_{\mathcal{Y}/\mathcal{X}}\longrightarrow T^1_{\mathcal{Y}/S})$ by (4.3), hence is identified with $N'_{\mathcal{Y}/\mathcal{X}}$. All arguments are equally valid in the analytic category. □

4.11. **The space M_i.** Let $i: Y\hookrightarrow X$ be the inclusion of a closed subvariety in an irreducible symplectic manifold. Then, as a consequence of [FK87], there is a universal deformation space M_i for locally trivial deformations of i, as a germ of complex spaces, see [Le11, VI.3]. The inclusion $Y\hookrightarrow X$ gives a point $0\in M_i$ and X determines a point $0\in M$ in the deformation space of X. By construction there is a forgetful morphism $p: M_i\longrightarrow M$ of complex spaces with $p(0)=0$. Let $R_X=\widehat{\mathcal{O}_{M,0}}$ and $R_i=\widehat{\mathcal{O}_{M_i,0}}$ be the completions at 0 and let $p^{\#}: R_X\longrightarrow R_i$ be the induced ring homomorphism. The following lemma is an immediate consequence of the universality of the deformations.

Lemma 4.12. The algebras R_i and R_X prorepresent D_i^{lt}, D_X so that

$$D_i^{\mathrm{lt}}=\mathrm{Hom}(R_i,\cdot)\quad\text{and}\quad D_X=\mathrm{Hom}(R_X,\cdot)$$

and the map of functors induces map $p^{\#}: R_X\longrightarrow R_i$.

4.13. **The T^1-lifting Principle.** To prove smoothness of M_i at 0 we will use Ran's T^1-lifting principle [Ra92Def]. Ran's ideas were developed further by Kawamata [Kaw92, Kaw97]. The method works in two steps.
The *first step* works for every prorepresentable deformation functor D, which has an obstruction space T^2. Put $A_n := k[t]/t^{n+1}$ and let $A_{n+1}\longrightarrow A_n$ be the canonical projection. To prove unobstructedness of D it suffices to show that the induced map $D(A_{n+1})\longrightarrow D(A_n)$ is always surjective by Lemma 1.5. However we want to replace this by a different criterion. Therefore we introduce the k-algebras $B_n := A_n[\varepsilon]$ and $C_n := A_n[\varepsilon]/\varepsilon t^n$. There are canonical projections $C_n\longrightarrow B_{n-1}$ and $B_n\longrightarrow C_n\longrightarrow A_n$. The last one is split by the inclusion $A_n\longrightarrow B_n$.

Lemma 4.14. Let $B_n\longrightarrow C_n$ be the canonical surjection. If the induced map $D(B_n)\longrightarrow D(C_n)$ is surjective, then $D(A_{n+1})\longrightarrow D(A_n)$ is surjective.

Proof. We have a morphism of small extensions in Art_k:

$$\begin{tikzcd} 0 \arrow[r] & (t^{n+1}) \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d, "\delta"] & A_n \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & (\varepsilon t^n) \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0 \end{tikzcd} \tag{4.5}$$

where $\delta(t)=t+\varepsilon$. The morphism $(t^{n+1})\longrightarrow(\varepsilon t^n)$ is multiplication by $n+1$ and hence an isomorphism as $\operatorname{char} k=0$. If we apply D to diagram (4.5),

we obtain

$$
\begin{tikzcd}
D(A_{n+1}) \arrow[r] \arrow[d, "\delta"] & D(A_n) \arrow[r] \arrow[d, "\delta"] & T^2 \otimes (t^{n+1}) \arrow[d, "\cong"] \\
D(B_n) \arrow[r] & D(C_n) \arrow[r] & T^2 \otimes (\varepsilon t^n)
\end{tikzcd}
\tag{4.6}
$$

Since $D(B_n) \longrightarrow D(C_n)$ is surjective, $D(C_n) \longrightarrow T^2 \otimes (\varepsilon t^n)$ is the zero map. The claim now follows by diagram chase. □

For an element $\xi_n \in D(A_n)$ we denote by $\xi_n|_{A_{n-1}}$ the image of ξ_n under the canonical map $D(A_n) \longrightarrow D(A_{n-1})$. Recall that $D(B_n)_{\xi_n} = \varphi_B^{-1}(\xi_n)$ where $\varphi_B : D(B_n) \longrightarrow D(A_n)$ is the canonical map.

Lemma 4.15. The morphism $D(B_n) \longrightarrow D(C_n)$ is surjective if for all $\xi_n \in D(A_n)$ and $\xi_{n-1} := \xi_n|_{A_{n-1}}$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

between the fibers over ξ_n and ξ_{n-1} is surjective.

Proof. To see this, we consider the diagram

$$
\begin{tikzcd}
D(B_n) \arrow[r] \arrow[d] \arrow[dd, bend right, "\chi"'] & D(A_n) \arrow[d, equal] \\
D(C_n) \arrow[r, "\varphi_C"] \arrow[d, "\psi"] & D(A_n) \arrow[d] \\
D(B_{n-1}) \arrow[r, "\varphi_B"] & D(A_{n-1})
\end{tikzcd}
\tag{4.7}
$$

where all morphisms are induced by the canonical projections, see section 4.13. Let $\eta \in D(C_n)$ be given and put $\xi_n := \varphi_C(\eta) \in D(A_n)$. The lower square is cocartesian, as D is prorepresentable and already the square of rings is cocartesian. Therefore the restriction of ψ to the fiber $D(C_n)_{\xi_n} = \varphi_C^{-1}(\xi_n)$ gives a bijection

$$D(C_n)_{\xi_n} \xrightarrow{\psi} D(B_{n-1})_{\xi_{n-1}}$$

onto the fiber over ξ_{n-1}. By assumption, $D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$ is surjective. Hence, there is $\eta' \in D(B_n)_{\xi_n}$ with $\chi(\eta') = \psi(\eta)$, so η' is a preimage of η and the claim follows. □

We summarize Lemma 1.5, Lemma 4.14 and Lemma 4.15 in

Lemma 4.16. Let D be a prorepresentable deformation functor, which has an obstruction space T^2. Then D is unobstructed if for all $\xi_n \in D(A_n)$ and $\xi_{n-1} := \xi_n|_{A_{n-1}}$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

is surjective. □

The *second step* of the T^1-lifting principle is to actually prove surjectivity of the map $D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$ for all ξ_n and ξ_{n-1} as in Lemma 4.16. This is not in general fulfilled and needs more input from the concrete geometric situation. We deduce this for $D = D_i^{\mathrm{lt}}$ from the fact that the sheaves Rg_*T_I from (1.8) are locally free and compatible with base change. Consider a simple normal crossing Lagrangian subvariety $i : Y \hookrightarrow X$ in an irreducible symplectic manifold X. Let $S = \operatorname{Spec} R$ for $R \in \mathrm{Art}_{\mathbb{C}}$ and let

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

be a locally trivial deformation of i over S. Consider the long exact sequence

$$0 \longrightarrow R^0g_*T_I \longrightarrow R^0g_*T_{\mathcal{X}/S} \longrightarrow R^0f_*N'_{\mathcal{Y}/\mathcal{X}} \longrightarrow R^1g_*T_I \longrightarrow \dots \tag{4.8}$$

obtained from the sequence (1.6). The symplectic form gives an isomorphism $T_{\mathcal{X}/S} \cong \Omega_{\mathcal{X}/S}$. By Lemma 4.7, $\mathcal{Y} \hookrightarrow \mathcal{X}$ is Lagrangian and hence by Proposition 4.10 we have $N'_{\mathcal{Y}/\mathcal{X}} \cong \widetilde{\Omega}_{\mathcal{Y}/S}$. Moreover, the module $R^0g_*\Omega_{\mathcal{X}/S}$ is free and compatible with base change by [Le12, Theorem 4.13]. This gives $R^0g_*\Omega_{\mathcal{X}/S} \otimes_R k = H^0(X, \Omega_X) = 0$, where the last equality holds as X is irreducible symplectic. By Nakayama's Lemma this implies $R^0g_*\Omega_{\mathcal{X}/S} = 0$. Put together this gives the following long exact sequence

$$0 \longrightarrow R^0f_*\widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow R^1g_*T_I \longrightarrow R^1g_*\Omega_{\mathcal{X}/S} \longrightarrow R^1f_*\widetilde{\Omega}_{\mathcal{Y}/S} \longrightarrow \dots \tag{4.9}$$

Lemma 4.17. Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in an irreducible symplectic manifold and let $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ be a locally trivial deformation of i over $S = \operatorname{Spec} R$ where $R \in \mathrm{Art}_{\mathbb{C}}$. Then the modules $R^kg_*T_I$ are free for all k and all morphisms in (4.9) have constant rank. In particular, all morphisms in (4.8) have constant rank.

Proof. By Theorem [Le12, Theorem 4.13] we know that $R^kg_*\Omega_{\mathcal{X}/S}$ is free. By Corollary 4.5 we know that Y is a projective variety, so Theorem [Le12, Theorem 4,13] applies and $R^kf_*\widetilde{\Omega}_{\mathcal{Y}/S}$ is free. Then by Theorem [Le12, Theorem 4.22] also the cokernel (and hence the kernel) of $R^kg_*\Omega_{\mathcal{X}/S} \longrightarrow R^kf_*\widetilde{\Omega}_{\mathcal{Y}/S}$ is free. So if we split the sequence (4.9) into pieces and use that

if $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ is exact and F', F'' are free, then so is F, we obtain freeness of $R^k g_* T_I$ for all k. □

Thus, the T^1-lifting principle may be applied.

Theorem 4.18. *Let Y be a Lagrangian simple normal crossing subvariety. Then the complex space M_i is smooth at 0.*

Proof. We put $D := D_i^{\mathrm{lt}}$ and denote by A_n, B_n and C_n the algebras introduced in section 4.13. For $\xi_n \in D(A_n)$ we put $\xi_{n-1} := \xi_n|_{A_{n-1}}$. By Lemma 4.16 the functor D is unobstructed if for all $\xi_n \in D(A_n)$ the map

$$D(B_n)_{\xi_n} \longrightarrow D(B_{n-1})_{\xi_{n-1}}$$

is surjective. For a given class $\xi_n \in D(A_n)$ take a deformation locally trivial

$$\begin{tikzcd} \mathcal{Y}_n \arrow[r, hook, "i_n"] \arrow[dr, "f"'] & \mathcal{X}_n \arrow[d, "g"] \\ & S_n \end{tikzcd}$$

of i over $S_n = \operatorname{Spec} A_n$ representing ξ_n. Let $i_{n-1} : \mathcal{Y}_{n-1} \hookrightarrow \mathcal{X}_{n-1}$ be the restriction of i_n to S_{n-1}. Then by Lemma 1.10 the diagram

$$\begin{tikzcd} R^1 g_* T_{i_n} \arrow[r] \arrow[d] & R^1 g_* T_{i_{n-1}} \arrow[d] \\ D_i(B_n)_{i_n} \arrow[r] & D_i(B_{n-1})_{i_{n-1}} \end{tikzcd}$$

is commutative and the vertical maps are bijections. By Lemma 4.17 the module $R^1 g_* T_{i_n}$ is free and hence by [EGAIII, Prop 7.8.5] it is compatible with base change. This means that $R^1 g_* T_{i_{n-1}} = R^1 g_* T_{i_n} \otimes_{A_n} A_{n-1}$. Clearly, $R^1 g_* T_{i_n} \longrightarrow R^1 g_* T_{i_n} \otimes_{A_n} A_{n-1}$ is surjective, which completes the proof. □

4.19. **Definition and Smoothness of M_Y.** By Theorem 4.18, the canonical morphism $p : (M_i, 0) \longrightarrow (M, 0)$ is just a holomorphic map between (germs of) complex manifolds, where $0 := [i : Y \hookrightarrow X] \in M_i$ and $0 := [X] \in M$ denote the distinguished points. We prove that its differential Dp has constant rank in a neighbourhood of 0. As an elementary consequence of the implicit function theorem, this implies that p is a submersion over its smooth image, see [Le11, Lem VI.4.2] for details.

Theorem 4.20. *Let $i : Y \hookrightarrow X$ be a Lagrangian simple normal crossing subvariety in an irreducible symplectic manifold X. Then there are open neighbourhoods $U \subset M_i$ of $0 \in M_i$ and $V \subset M$ of $0 \in M$ such that $M_Y := p(U) \subset V$ is a closed submanifold and $p : U \longrightarrow M_Y$ is a smooth morphism.*

Proof. By Theorem 4.18 and the Bogomolov-Tian-Todorov theorem we know that M_i and M are smooth at 0. By the implicit function theorem we have to show that the differential Dp of $p : M_i \longrightarrow M$ has constant rank in a neighborhood of 0. This holds if the stalk of $\operatorname{coker}(p_* : T_{M_i} \longrightarrow p^*T_M)$ at 0 is free. Freeness may be tested after completion, so we have to verify that $p_* : T_{R_i} \longrightarrow T_{R_X}$ has constant rank, where $R_X = \widehat{\mathcal{O}_{M,0}}$ and $R_i = \widehat{\mathcal{O}_{M_i,0}}$, compare to Lemma 4.12. By the local criterion for flatness [Ser06, Thm A.5] this follows if

$$T_{R_i} \otimes_{R_i} R_i/\mathfrak{m}_i^n \longrightarrow T_{R_X} \otimes_{R_X} R_i/\mathfrak{m}_i^n \tag{4.10}$$

has constant rank for all n. Let $\eta : R_i \longrightarrow A$ be a $\mathbb{C}$-algebra homomorphism corresponding to a locally trivial deformation

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "f"'] & \mathcal{X} \arrow[d, "g"] \\ & S \end{tikzcd}$$

of i over $S = \operatorname{Spec} A$ and let $q : A[\varepsilon] \longrightarrow A$ be given by $\varepsilon \mapsto 0$. Then

$$\begin{aligned} D_i^{\mathrm{lt}}(A[\varepsilon])_\eta = \operatorname{Hom}(R_i, A[\varepsilon])_\eta = \operatorname{Der}_{\mathbb{C}}(R_i, A) = \operatorname{Hom}_{R_i}(\Omega_{R_i/k}, A) \\ = T_{R_i} \otimes_{R_i} A \end{aligned}$$

where $\operatorname{Hom}(R_i, A[\varepsilon])_\eta = \{\varphi \in \operatorname{Hom}(R_i, A[\varepsilon]) \mid q \circ \varphi = \eta\}$. Similarly, we find that $D_X(A[\varepsilon])_\xi = T_{R_X} \otimes_{R_X} A$ for $\xi : R_X \longrightarrow A$. Now let $A = R_i/\mathfrak{m}_i^n$, let $\eta : R_i \longrightarrow R_i/\mathfrak{m}_i^n$ be the canonical projection and let $\xi = \eta \circ p^\#$ where $p^\# : R_X \longrightarrow R_i$ is the canonical map. Furthermore, we have $D_i^{\mathrm{lt}}(A[\varepsilon])_\eta = R^1 g_* T_I$ and $D_X(A[\varepsilon])_\xi = R^1 g_* T_{\mathcal{X}/S}$ by Lemma 1.10 and Lemma 1.7. Moreover, the map (4.10) is identified with $R^1 g_* T_I \longrightarrow R^1 g_* T_{\mathcal{X}/S}$ from (4.8), which is of constant rank by Lemma 4.17. This completes the proof. $\square$

5. Main results

Let $i : Y \hookrightarrow X$ be the inclusion of a simple normal crossing Lagrangian subvariety in an irreducible symplectic manifold. We denote by $\nu : \widetilde{Y} \longrightarrow Y$ the normalization and by $j = i \circ \nu$ the composition. We will compare the space $M_Y = \operatorname{im}(M_i \longrightarrow M)$ as defined in Theorem 4.20 with the spaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ from section 2.

Lemma 5.1. Suppose Y has simple normal crossings. Then

$$\ker\left(j^* : H^1(\Omega_X) \longrightarrow H^1(\Omega_{\widetilde{Y}})\right) = \ker\left(i^* : H^1(\Omega_X) \longrightarrow H^1(\widetilde{\Omega}_Y)\right),$$

where $\nu : \widetilde{Y} \longrightarrow Y$ is the normalization.

Proof. As $j^* = \nu^* \circ i^*$ the inclusion $\supset$ is obvious. For the other direction it suffices to show that ν^* is injective on $\operatorname{im} i^*$. By Corollary 4.5 the subvariety Y is projective, hence by [Del71, Del74] there is a functorial mixed Hodge structure on $H^k_Y := H^k(Y, \mathbb{C})$ for every k. We denote by $F^\bullet$ the Hodge filtration on H^2_Y and by $W_\bullet$ the weight filtration on H^2_Y. As a special case of [Le12, Cor 4.16] we deduce that

$$H^1(\widetilde{\Omega}_Y) = \mathrm{Gr}^1_F H^2_Y = F^1 H^2_Y / F^2 H^2_Y.$$

Let $\ldots \substack{\rightarrow\\[-1em]\rightarrow\\[-1em]\rightarrow} Y^1 \rightrightarrows Y^0 \to Y$ be the canonical semi-simplicial resolution, see e.g. [Le12, 4.8]. Note that $\widetilde{Y} = Y^0$. Consider the weight spectral sequence associated to the first graded objects of the Hodge filtration given by

$$E_1^{r,s} = H^s(Y^r, \Omega^1_{Y^r}) \Rightarrow H^{r+s}(Y, \widetilde{\Omega}^1_Y) \tag{5.1}$$

By [PS08, Thm 3.12 (3)] it degenerates on E_r if the weight spectral sequence degenerates at E_r. But the latter is known to degenerate at E_2. The differential $d_1 : E_1^{0,1} \longrightarrow E_1^{0,1}$ is given by $\delta : H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1})$ and degeneration at E_2 tells us that

$$\begin{aligned}\mathrm{Gr}^W_2 \mathrm{Gr}^1_F H^2_Y &= F^1 H^2_Y / (W_1 F^1 H^2_Y + F^2 H^2_Y) = E_\infty^{0,1} = E_2^{0,1} \\ &= \ker\left(H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1})\right).\end{aligned}$$

In other words, as $W_2 \mathrm{Gr}^1_F H^2_Y = \mathrm{Gr}^1_F H^2_Y = H^1(\widetilde{\Omega}_Y)$ there is an exact sequence

$$0 \longrightarrow W_1 \mathrm{Gr}^1_F H^2_Y \longrightarrow H^1(\widetilde{\Omega}_Y) \xrightarrow{\nu^*} H^1(\Omega_{Y^0}) \longrightarrow H^1(\Omega_{Y^1}),$$

so that $\ker \nu^* = W_1 \mathrm{Gr}^1_F H^2_Y$. But $H^2_X := H^2(X, \mathbb{C})$ has pure weight two because X is smooth. In particular, $W_1 \mathrm{Gr}^1_F H^2_X = 0$. Morphisms of mixed Hodge structures are strict with respect to both filtrations, so we have

$$0 = i^*(W_1 \mathrm{Gr}^1_F H^2_X) = \operatorname{im} i^* \cap W_1 \mathrm{Gr}^1_F H^2_Y = \operatorname{im} i^* \cap \ker \nu^*$$

hence ν^* is injective on $\operatorname{im} i^*$ and we deduce $\ker i^* = \ker j^*$ completing the proof. $\square$

The following lemma generalizes [Vo92, Lem 2.3] to the normal crossing case.

Lemma 5.2. Suppose Y has simple normal crossings. Then we have $T_{M'_Y,0} = T_{M_Y,0}$ for the Zariski tangent spaces at $0 \in M_Y \cap M'_Y$.

Proof. By Lemma 3.3 the tangent space of M'_Y at 0 is

$$T_{M'_Y,0} = \ker\left(j^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\widetilde{\Omega}_Y)\right).$$

By Lemma 5.1 we have

$$T_{M'_Y,0} = \ker\left(i^* \circ \omega' : H^1(X, T_X) \longrightarrow H^1(\Omega_{\widetilde{Y}})\right),$$

where $\widetilde{Y} \longrightarrow Y$ is the normalization. On the other hand, M_Y is the smooth image of $p : M_i \longrightarrow M$ so that

$$\begin{aligned} T_{M_Y,0} &= \operatorname{im}\left(p_* : T_{M_i,0} \longrightarrow T_{M,0}\right) \\ &= \operatorname{im}\left(H^1(X, T_i) \longrightarrow H^1(X, T_X)\right) \\ &= \ker\left(H^1(X, T_X) \xrightarrow{\alpha} H^1(Y, N'_{Y/X})\right) \end{aligned}$$

where the third equality holds because the sequence (4.8) is exact.
By (4.3) and Proposition 4.10 we have a commutative diagram

$$\begin{tikzcd} H^1(X, \Omega_X) \arrow[r, "j^*"] & H^1(Y, \widetilde{\Omega}_Y) \arrow[d, "\tilde{\omega}"] \\ H^1(X, T_X) \arrow[u, "\omega'"] \arrow[r, "\alpha"] & H^1(Y, N'_{Y/X}) \end{tikzcd}$$

where the vertical maps are isomorphisms. This implies that

$$T_{M_Y,0} = \ker(\alpha) = \ker(\tilde{\omega} \circ j^* \circ \omega') = \ker(j^* \circ \omega') = T_{M'_Y,0}$$

and completes the proof. $\square$

Theorem 5.3. *Let $i : Y \hookrightarrow X$ be a simple normal crossing Lagrangian subvariety in a compact irreducible symplectic manifold X, let $\nu : \widetilde{Y} \longrightarrow Y$ be the normalization and denote $j = i \circ \nu$. Then $M'_Y = M_Y$ and this space is smooth at 0 of codimension*

$$\operatorname{codim}_M M_Y = \operatorname{codim}_M M'_Y = \operatorname{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right) \tag{5.2}$$

in M.

Proof. Assume that $Y = \cup_i Y_i$ is a decomposition into irreducible components. In section 2.4 we defined the subspaces M'_Y, $M'_{[Y]}$ and $M_{[Y]}$ of M associated to a Lagrangian subvariety Y of X. We have

$$\begin{tikzcd} M'_Y \arrow[r, hook, dashed] \arrow[d, equal] & M'_{[Y]} \arrow[r, equal] & M_{[Y]} \\ \bigcap_i M'_{Y_i} \arrow[r, equal] & \bigcap_i M'_{[Y_i]} \arrow[u, hook] \arrow[r, equal] & \bigcap_i M_{[Y_i]} \arrow[u, hook] \end{tikzcd} \tag{5.3}$$

where the vertical relations were observed in Remark 2.7, the horizontal equalities on the right were shown in Proposition 3.2 and the left lower

equality holds as Y has simple normal crossings by Corollary 3.5. As a consequence, we obtain the upper left inclusion.
As a consequence of [Le12, Lemma 4.5] we have $M_Y \subset \cap_i M_{Y_i}$. As M_{Y_i} is smooth, in particular reduced, for each i, we have that $M_{Y_i} \subset M_{[Y_i]}$ so that

$$M_Y \subset \bigcap_i M_{Y_i} \subset \bigcap_i M_{[Y_i]} = M'_Y.$$

Therefore, we find

$$\dim M_Y \leq \dim M'_Y \leq \dim T_{M'_Y,0} = \dim T_{M_Y,0}$$

where the last equality comes from Lemma 5.2. As M_Y is smooth by Theorem 4.20, we have equality everywhere so that $M_Y = M'_Y$.
The statement about the codimension follows from the description (3.1) of the tangent space of M'_Y. □

Corollary 5.4. Let $K := \ker\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right)$, let q be the Beauville-Bogomolov quadratic form and consider the period domain

$$Q := \{\alpha \in \mathbb{P}(H^2(X,\mathbb{C})) \mid q(\alpha) = 0, q(\alpha + \bar{\alpha}) > 0\}$$

of X. Then the period map $\wp : M \longrightarrow Q$ identifies M_Y with $\mathbb{P}(K) \cap Q$ locally at $[X] \in M$.

Proof. As the period map identifies M with Q it suffices to show that $\wp(M_Y) = \mathbb{P}(K) \cap Q$. By [Huy99, 1.14], $\mathbb{P}(K) \cap Q$ is the locus where $K^\perp \subset H^2(X,\mathbb{C})$ remains of type $(1,1)$ and its codimension is $\dim K^\perp$. Note that $K^\perp \subset H^{1,1}(X)$ is defined over $\mathbb{Z}$ and therefore is spanned by the Chern classes of a collection of line bundles on X. By Lemma 4.7 the subspace $K^\perp$ remains of type $(1,1)$ over M_Y. Hence $\wp(M_Y) \subset \mathbb{P}(K) \cap Q$. Moreover, we have

$$\begin{aligned} \operatorname{codim}_Q \wp(M_Y) &= \operatorname{codim}_M M_Y = \operatorname{rk}\left(j^* : H^2(X,\mathbb{C}) \longrightarrow H^2(\widetilde{Y},\mathbb{C})\right) \\ &= b_2(X) - \dim K = \dim K^\perp \\ &= \operatorname{codim}_Q \mathbb{P}(K) \cap Q \end{aligned}$$

So both sets are equal. □

6. Applications to Lagrangian fibrations

In this section we give some applications of Theorem 5.3 to Lagrangian fibrations. Our main goal is to determine $\operatorname{codim}_M M_Y$. We show first that if we deform a fiber of a fibration then also the fibration deforms, see Lemma 6.4. We also pose a number of interesting questions regarding singular fibers, which hopefully contribute to understanding Lagrangian fibrations.
Recall the important

Theorem 6.1 (**Matsushita**). *Let X be an irreducible symplectic manifold of dimension $2n$. If B is a normal projective variety with $0 < \dim B < 2n$ and $f : X \longrightarrow B$ is a surjective morphism with connected fibers, then:*
$\dim B = n$, $-K_B$ is ample, the Picard number $\varrho(B)$ is one, f is equidimensional and every irreducible component of the reduction of a fiber is a Lagrangian subvariety.

In particular, if B is smooth, then f is flat by equidimensionality, see e.g. [Eis95, Thm 18.16]. Here, a singular variety is said to be *Lagrangian* if its regular part is Lagrangian in the ordinary sense. Such f as in the theorem is called a *Lagrangian fibration*. The theorem was proven in a series of papers, see [Mat99, Mat00, Mat01, Mat03]. The holomorphic Liouville-Arnol'd theorem shows that every smooth fiber is a complex torus, thence singular fibers enter the focus.
We want to apply Theorem 5.3 to singular fibers of Lagrangian fibrations. This would tell us, to where in M the fiber deforms as a subvariety. The following lemmas show that if the singular fiber deforms, then the fibration deforms and the deformation of the fiber remains vertical.

Lemma 6.2. Suppose we are given a diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, hook, "I"] \arrow[dr, "p"'] & \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi"] & P \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where S is an irreducible complex space, $\mathcal{X} \longrightarrow S$ is a proper family of irreducible symplectic manifolds, $\mathcal{Y} \longrightarrow S$ is a proper family of Lagrangian subvarieties and q and F are proper morphisms of complex spaces. Assume that for every $s \in S$ the morphism $F_s : \mathcal{X}_s \longrightarrow P_s$ obtained by base change is a Lagrangian fibration. If $\mathcal{Y} \longrightarrow S$ has connected fibers and if $F(\mathcal{Y}_0)$ is a point for some $0 \in S$, then also $F(\mathcal{Y}_s)$ is a point for all $s \in S$.

Proof. By Theorem 6.1 a Lagrangian fibration is equidimensional. Then the Lemma is just a special case of the Rigidity Lemma [KM98, Lem 1.6]. □

6.3. **Deforming fibrations.** Let $f : X \longrightarrow B$ Lagrangian fibration and assume that B is projective. Matsushita showed in [Mat09, Prop 2.1] that there is a smooth hypersurface $M_f \subset M$ with a relative Lagrangian fibration

extending f

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "F"] \arrow[dr, "\pi"'] & & P \arrow[dl] \\ & M_f & \end{tikzcd}$$

where $\pi : \mathfrak{X} \longrightarrow M_f$ is the restriction of the universal family to M_f and $P \longrightarrow M_f$ is a projective morphism. In particular, $F_t : \mathfrak{X}_t \longrightarrow P_t$ is a Lagrangian fibration and $F_0 = f$.

Let T be a smooth fiber of f and let $M_T \subset M$ be as in Theorem 4.20. Then clearly, $M_f \subset M_T$. By Voisin's theorem, M_T is smooth of codimension equal to $\operatorname{rk}\big(i^* : H^2(X, \mathbb{C}) \longrightarrow H^2(T, \mathbb{C})\big)$, where $i : T \hookrightarrow X$ is the inclusion. This rank is certainly ≥ 1, as the Kähler class restricts to a non-trivial element. We deduce that $M_T = M_f$ is a smooth hypersurface in M.

The following lemma tells us that if the reduced fiber is preserved as a subvariety, then also the fibration is preserved.

Lemma 6.4. Let $f : X \longrightarrow B$ be a Lagrangian fibration, let $t \in B$ and assume that $Y = (X_t)_{\text{red}}$ is a simple normal crossing Lagrangian subvariety. Then we have $M_Y \subset M_f$. Moreover, locally trivial deformations of Y remain fiber components.

Proof. By 6.3 it is sufficient to show $M_Y \subset M_T$. Let $Y = \cup_{i\in I} Y_i$ be a decomposition into irreducible components. By [Le12, Lemma 4.5], we have $M_Y \subset \cap_i M_{[Y_i]}$ and by Proposition 3.2 also $M_Y \subset \cap_i M'_{[Y_i]}$. But for a smooth fiber T of f we have $\sum_i n_i[Y_i] = [T]$ and so

$$\cap_i M'_{[Y_i]} \subset M'_{[\sum_i n_i Y_i]} = M'_{[T]} = M_T,$$

where the first two relations follow directly from Definition 2.6, the third equality is Voisin's theorem. Put together this gives $M_Y \subset M_T = M_f$. The last claim follows from Lemma 6.2. $\square$

6.5. **Codimension estimates.** Let X be an irreducible symplectic manifold and let $f : X \longrightarrow \mathbb{P}^n$ be a Lagrangian fibration. In view of [Hwa08, Thm 1.2], it seems reasonable to assume $\mathbb{P}^n$ to be the base of the fibration. We put $Y = (X_t)_{\text{red}}$ for $t \in D := \{t \in \mathbb{P}^n : X_t \text{ is singular}\}$. The analytic subset D is called the *discriminant locus* of f. We know by [Hwa08, Prop 4.1] and [HO09, Prop 3.1] that D is nonempty and of pure codimension one.

Let M be the universal deformation space of X and M_Y for its subspace from Theorem 5.3. Determining $\operatorname{codim}_M M_Y$ is interesting for several reasons. For example, there are several results assuming the general singular fibers to be of a special kind, see [HO10], [Saw08b], [Saw08a], [Thi08]. If we knew

that complicated general singular fibers only show up in higher codimension in M, we could always deform to such special situations.

Let $D_0 \ni t$ be an irreducible component of D and let $X_0 := X \times_B D_0 = f^{-1}(D_0)$. Let $Y = \cup_{i \in I} Y_i$ and $X_0 = \cup_{j \in J} X_j$ be decompositions into irreducible components and consider the surjective map $j : I \longrightarrow J$ mapping $i \in I$ to the unique $j = j(i) \in J$ with $Y_i \subset X_j$.

I am very grateful to Keiji Oguiso for explaining the following lemma.

Lemma 6.6. Let $f : X \longrightarrow \mathbb{P}^n$ be a Lagrangian fibration of a projective irreducible symplectic manifold X. Let $X_0 = \bigcup_{j \in J} X_j$ where $J = \{1, \ldots, r\}$ and let $i : Y = (X_t)_{\mathrm{red}} \hookrightarrow X$ for $t \in D_0 \subset \mathbb{P}^n$ be the reduction of a general singular fiber contained in X_0. Then

$$\mathrm{rk}\left(j^* : H^2(X, \mathbb{C}) \longrightarrow H^2(\widetilde{Y}, \mathbb{C})\right) \geq r,$$

where $\nu : \widetilde{Y} \longrightarrow Y$ is the normalization and $j = \nu \circ i$. More precisely, the subspace of $H^2(X, \mathbb{C})$ generated by the classes of the divisors X_j maps onto a subspace of of dimension $\geq r - 1$ not containing the class of the ample divisor.

Proof. If we take a general line $\ell \subset \mathbb{P}^n$, then the fiber product $X_\ell = X \times_{\mathbb{P}^n} \ell$ is smooth by Kleiman's theorem [Kle74, 2. Thm]. As $t \in D_0$ is general, there is such a line with $t \in \ell$. Let H be a very ample divisor on X and let $H_1, \ldots, H_{n-1} \in |H|$ be general. Then the intersection $S = X_\ell \cap H_1 \cap \ldots \cap H_{n-1}$ is a smooth surface by Bertini's theorem. By construction it comes with a morphism $g : S \longrightarrow \mathbb{P}^1 \cong \ell$.

Consider the diagram

$$\begin{tikzcd} H^2(X, \mathbb{C}) \arrow[r, "j^*"] \arrow[d, "\varrho"'] & H^2(\widetilde{Y}, \mathbb{C}) \arrow[d, "\varrho_Y"] \\ H^2(S, \mathbb{C}) \arrow[r, "j_S^*"] & H^2(\widetilde{F}, \mathbb{C}) \end{tikzcd} \tag{6.1}$$

where $F = Y \cap H_1 \cap \ldots \cap H_{n-1} \subset S$ and $\widetilde{F} \longrightarrow F$ is the normalization. Note that $\widetilde{Y}$ is smooth by [HO09, Thm 1.3] and $\widetilde{F}$ is smooth, as F is a curve. Let $Y = \bigcup_{i=1}^s Y_i$ and $F = \bigcup_{\lambda=1}^q F_\lambda$ be decompositions into irreducible components where $s = \#I$. We put $F(i) := Y_i \cap H_1 \cap \ldots \cap H_{n-1} = \bigcup_{\lambda \in \Lambda_i} F_\lambda$, where $\Lambda_i \subset \Lambda := \{1, \ldots, q\}$ is the subset of all λ such that $F_\lambda \subset Y_i$. If the H_k are general enough, the irreducible components F_λ of $F(i)$ are mutually distinct for all i. In other words, Λ is the disjoint union of the Λ_i. Indeed, one only has to verify that no irreducible component of $Y_i \cap Y_j \cap H_1 \ldots \cap H_{k-1}$ is contained in H_k for all i, j, and k.

We will show that the subspace $V \subset H^2(X, \mathbb{C})$ spanned by the X_j and H maps surjectively onto an r-dimensional subspace in $H^2(\widetilde{F}, \mathbb{C})$. This would imply the claim by diagram (6.1).

Let $n_j \in \mathbb{N}$ be the multiplicity of $X_0 = f^{-1}(D_0)$ along X_j. Then

$$X_0 = \sum_j n_j X_j \quad \text{and} \quad X_t = \sum_i n_{j(i)} Y_i$$

as cycles, where as above $j(i)$ is the unique $j \in J$ with $Y_i \subset X_j$. Recall that $\Lambda = \coprod_i \Lambda_i$ is a disjoint union. So $n_\lambda := n_{j(i)}$ for $\lambda \in \Lambda_i$ is well-defined and we have $F = \sum_\lambda n_\lambda F_\lambda$. As $F = \bigcup_{\lambda=1}^q F_\lambda$ we obtain $\widetilde{F} = \bigcup_{\lambda=1}^q \widetilde{F}_\lambda$ where $\widetilde{F}_\lambda$ is the normalization of F_λ. Thus,

$$H^2(\widetilde{F}, \mathbb{C}) \cong \bigoplus_{\lambda=1}^q H^2(\widetilde{F}_\lambda, \mathbb{C}) \cong \mathbb{C}^q.$$

If we denote the intersection pairing on S by $(\cdot, \cdot)_S$, then under this isomorphism $j_S^* : H^2(S, \mathbb{C}) \longrightarrow H^2(\widetilde{F}, \mathbb{C})$ is given by

$$\alpha \mapsto ((\alpha, F_1)_S, \ldots, (\alpha, F_q)_S).$$

Let $\{x_\lambda \mid \lambda \in \Lambda\} \subset H^2(\widetilde{F}, \mathbb{C})^\vee$ be the dual basis of the basis of $H^2(\widetilde{F}, \mathbb{C})$ obtained corresponding to the standard basis of $\mathbb{C}^q \cong H^2(\widetilde{F}, \mathbb{C})$. By Zariski's Lemma [BHPV, Ch III, Lem 8.2] the subspace $W \subset H^2(S, \mathbb{C})$ spanned by the classes of the F_λ maps surjectively to the hyperplane of $\mathbb{C}^q$ given by $\sum_\lambda n_\lambda x_\lambda = 0$, So the subspace of $H^2(S, \mathbb{C})$ spanned by the classes of the F_λ and $H|_S$ maps surjectively onto $\mathbb{C}^q$. We have $\varrho_Y(j^* X_j) = j_S^* \varrho(X_j) = \big((\varrho(X_j), F_\lambda)_S\big)_\lambda$. As the Λ_i are mutually disjoint, so are the $\Lambda_j := \bigcup_{j(i)=j} \Lambda_i$. We see from $(\varrho(X_j), F_\lambda)_S = \sum_{\mu \in \Lambda_j} (F_\mu, F_\lambda)_S$ that the subspace of $H^2(X, \mathbb{C})$ generated by the X_j surjects onto a subspace of $\mathbb{C}^q$ of dimension $\geq r - 1$. The claim follows as the image of V does not contain $j_S^*(H|_S)$. □

For $K \subset I$ let $Y_K := \bigcup_{i \in K} Y_i$ and let $r_K := |\{j(i) \mid i \in K\}|$. We obviously have $r_K \leq r_I = r$.

Corollary 6.7. With the notation above,

$$\begin{aligned} \operatorname{codim} M_Y &\geq r \\ \operatorname{codim} M_{Y_K} &\geq r_K \quad \text{and} \quad \geq r_K + 1 \quad \text{if} \quad Y_K \neq Y. \end{aligned}$$

Proof. This follows from Theorem 5.3 and Lemma 6.6. For the last statement one uses that by Zariski's Lemma the map j_S^* from the proof of Lemma 6.6 is surjective if $Y_K \neq Y$. □

In view of Lemma 6.6 it seems that the codimension of M_Y is rather influenced by the number of irreducible components of $X_0 = f^{-1}(D_0)$ than by

the number of irreducible components of Y. Thus, a very interesting and important question is the following

Question 6.8. Let $Y = \cup_{i\in I} Y_i$ and $X_0 = \cup_{j\in J} X_j$ as in the beginning of section 6.5. Is then $\#I = \#J$? Do we always have $\operatorname{codim}_M M_Y = \#J$ for simple normal crossing Y?

There is no obvious reason, why these numbers should be equal, but in all examples we know they are equal.

6.9. **Vista.** As our main results are built from many pieces, there is obviously ample space for generalizations. First of all, Theorem 5.3 should be true literally for normal crossing singularities. We can proof this in all relevant examples, see for instance Example 6.10 below and [Le11] for more details.

Example 6.10. Let Y be a normal crossing variety, which is obtained by identifying two disjoint sections of a $\mathbb{P}^1$-bundle over an abelian variety, possibly along a translation. If Y is a Lagrangian subvariety of an irreducible symplectic manifold X one can prove the analogues of Theorem 4.18, Theorem 4.20 and Theorem 5.3 . In particular, $\operatorname{codim}_M M_Y = \operatorname{rk}\big(H^2(X,\mathbb{C}) \longrightarrow H^2(Z,\mathbb{C})\big)$, see [Le11, Example VII.2.4].
Indeed, such varieties show up as singular fibers of Lagrangian fibrations on irreducible symplectic manifolds, see [Bea99, 1.2] or [Saw08b, 2.]. Therefore, they persist whenever a smooth fiber persists, as $M_Y = M_{[Y]}$ and $[Y]$ coincides with the class of a smooth fiber. In particular, $\operatorname{codim}_M M_Y = 1$.

This example leads directily to the task of determining the singular fibers, that show up generically in the moduli space. We pose

Question 6.11. For which of the general singular fibers Y of Hwang-Oguiso [HO09] is $\operatorname{codim}_M M_Y = 1$? Note that as $\operatorname{codim} M_f = 1$ and as there are always singular fibers, there have to be fibers with this property.

In the case of K3 surfaces, the situation becomes easier. For elliptic K3 surfaces it was shown in [Le11, Thm VII.3.8] that $\operatorname{codim} M_Y$ is equal to the number $\#I = \#J$ of irreducible components of the reduced fiber, if the latter has normal crossings, and $\operatorname{codim} M_Y \geq \#I$ in all other cases.
Our results will definitely not carry over literally to all kinds of singularities. For example, for a cuspidal rational curve Y in a K3 surface we have $M_Y \subsetneq M'_Y$.

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